Indian Statistical Institute, Bangalore
B. Math II, First Semester, 2024-25
Mid-semester Examination, Introduction to Statistical Inference
11.09.24 Maximum Score 100 Duration: 3 Hours

1. (10) Let $\mathbf{X}_1, \cdots \mathbf{X}_n$ be i.i.d. $\mathcal{N}(\mu, \sigma^2)$ random variables. Show that $(\bar{\mathbf{X}}, s^2)$ is jointly sufficient for (μ, σ^2), where
$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (\mathbf{X}_i - \bar{\mathbf{X}})^2.$$

2. (15) Suppose $\delta(X)$ is a Bayes estimator of parameter $g(\theta)$ under prior π. Suppose $\delta(X)$ is also unbiased for $g(\theta)$.
   (a) Show, by conditioning on X, that $E(\delta(X)g(\theta)) = E(\delta^2(X))$.
   (b) Show, by conditioning on θ, that $E(\delta(X)g(\theta)) = E(g^2(\theta))$.
   (c) Conclude that $E(\delta(X) - g(\theta))^2 = 0$.
   (d) If $X_1, \cdots, X_n$ are iid $\mathcal{N}(\mu, 1)$, then use the above result to conclude that $\bar{X}$ cannot be a Bayes estimator for μ under any prior.
   (e) What can you conclude about unbiased Bayes estimators?

3. (20) Consider the regression model:
$$y_i = bx_i + e_i, 1 \le i \le n,$$
where x_i's are fixed non-zero real numbers and e_i's are independent random variables with mean zero and equal variance.
   (a) Find the least squares estimator of b.
   (b) Show that an estimator of the form $\sum_{i=1}^{n} a_i y_i$ (where a_i 's are non random real numbers) is unbiased for b iff $\sum_{i=1}^{n} a_i x_i = 1$.
   (c) Show that the least squares estimator has the minimum variance in the class of unbiased estimators.

4. (10) Suppose $X_1, \cdots, X_n$ is a simple random sample drawn without replacement from a population $y_1, \cdots, y_N$. Find the mean and variance of $\bar{X}$ in terms of the population mean, variance, n and N.

5. (15) Let X_1, X_2 be independent $\mathcal{N}(\theta, 1)$ random variables. Then $\bar{X}$ is unbiased for θ. Let $T = E(\bar{X} \mid X_1)$.
   (a) Show that $E(T) = \theta$.
   (b) Show that the variance of T is lower than that of $\bar{X}$.
   (c) Is T better than $\bar{X}$ as an estimator for θ? Justify your answer.

6. (15) Let $X_1, \cdots, X_n$ be iid Geometric(p) random variables, each X_i denoting the number of Bernoulli(p) trials required for the first success.
   (a) Show that $\bar{X}$ attains the Cramer-Rao lower bound.
   (b) Hence conclude that $\bar{X}$ is UMVUE for $1/p$.

7. (15) Consider the bivariate normal density
$$h(x, y) = \frac{1}{2\pi} \exp\left\{-\frac{1}{2}\left(2x^2 + y^2 + 2xy - 22x - 14y + 65\right)\right\}$$
Find the mean vector and the covariance matrix.